Pergamon

Journal of Structural Geology, Vol. 20, No. 12, pp. 1783 to 1793, 1998

PII: S0191-8141(98)00071-6
0191-8141/98 $ - see front matter

Toward more realistic formulations for the analysis of laccoliths

ARNOLD D. KERR

Department of Civil and Environmental Engineering, University of Delaware, Newark, DE 19716, U.S.A.

and

DAVID D. POLLARD

Department of Geological and Environmental Sciences, Stanford University, Stanford, CA 94305, U.S.A.

(*Received* 28 *August* 1996; *accepted in revised form* 15 *May* 1998)

Abstract—The published laccolith analyses are based on the linear plate bending theory and the *a priori* assumption that the width of the laccolith is fixed. This is not the case in an actual situation. The dimension of the laccolith in the horizontal plane has to result from an additional matching condition at the separation lines. The published analyses are generalized by dropping the *a priori* assumption that the width of the laccolith is prescribed, by assuming that the magmatic pressure is not constant, and by taking into consideration the vertical compressibility of the overburden "plate" and base in the contact region. In order to determine the magnitude of the magmatic pressure, a condition is postulated that equates the measured volume of the intruded magma in a laccolith with the corresponding analytical expression for the volume. The obtained closed-form solution appears to satisfy many of the intuitive expectations. It was evaluated numerically and the results are presented as graphs. It may be concluded that even very small laccoliths may exist, provided the magmatic pressure is sufficiently larger than the overburden weight. We also show the dependence of the laccolith size on its stratigraphic position; the thicker the overburden h the larger the size of the laccolith, for an overburden plate of given thickness, the larger the volume V of the intruded magma, the larger the laccolith width $2a$ and its height. The paper concludes by discussing a published analysis for laccolith with flexible underburden and overburden. It is shown that this analysis is based on a formulation that is of questionable validity.

INTRODUCTION

During the past three decades a number of analyses for the laccolith problem have been published which are based on the linear plate theory of Timoshenko and Woinowsky-Krieger, 1959 (e.g., Johnson, 1970; Pollard and Johnson, 1973). The approach adopted in these analyses seems to be accepted generally, as indicated by the laccolith presentation by Turcotte and Schubert (1982). For recent discussions of the geological and geophysical aspects of the laccolith problems refer to Corry (1988) and Jackson and Pollard (1988, 1990).

The fundamental concept of laccolithic intrusions was conceived by G. K. Gilbert during his geological investigations of the Henry Mountains, Utah, near the end of the 19th Century (Gilbert, 1877). Although the facts that igneous rocks, once molten, may erupt to form volcanoes were understood at that time, little was known about how *magma* traveled from the site where it melted (presumably in the upper mantle or lower crust) upward toward the Earth's surface. Gilbert inferred from his field observations that the magma intruded upward through the horizontal sedimentary rocks of the Colorado Plateau in tabular dikes or narrow pipe-like conduits. At depths of a few kilometers below the Earth's surface the magma turned and intruded laterally along bedding planes, forming horizontal, tabular *sills*. For example, in the Henry Mountains these sills range from a few meters to about 20 m in thickness. Some of these sills attained sufficient area to begin lifting and bending the overlying sedimentary strata and thus began the formation of *laccoliths*. According to Jackson and Pollard (1988), the transition between sill and laccolith began when the magma spread to a diameter of about 2–6 km. Laccoliths are characterized by roughly flat bottom contacts, conformable with the underlying sedimentary host rock, and arched upper contacts, conformable with overlying domes of sedimentary strata.

In the southern Henry Mountains the major *laccoliths* at Mount Holmes, Mount Ellsworth, and Mount Hillers are roughly circular in plan form, with diameters of 10–14 km (Jackson and Pollard 1988, figs 7–9). Their bases were at a maximum depth of about 4 km at the time of intrusion and the sedimentary strata was uplifted about 1.2, 1.8 and at least 2.5 km during emplacement of the magma to form the sedimentary domes that compose these three mountains.

Plate bending theory has been used to investigate the upward deflections of the overlying strata as driven by the magma pressure and resisted by the weight of the strata and its elastic stiffness. In most engineering applications of this theory, the ratio of plate thickness to length is less than 1/10, in keeping with the assumptions used to derive the governing equation for the vertical deflections (Timoshenko and Woinowsky-Krieger, 1959). For laccoliths, the ratio of overburden thickness to intrusive diameter is rarely that small. For example, in the southern Henry Mountains this ratio ranges from 2/7 to 2/5. However, as Pollard and Johnson (1973, figs 5 and 6) pointed out, the overburden apparently behaved mechanically as a stack of plates with variable properties, some of which slipped over one another during bending. Assuming that the shear stress between them is negligible, the *effective bending stiffness of this stack* is the sum of the stiffnesses of the individual plates. It is thus much smaller than the stiffness of a continuous plate. In this case each plate of the overburden with its own (much smaller) thickness may satisfy the 1/10 requirement, thereby bringing the overburden into the appropriate range for the application of plate theory. Jackson and Pollard (1990, figs 2–4) have identified and documented the bedding-plane faults that acted to decouple the strata and reduce the effective bending stiffness. To this stiffness there corresponds a *reduced effective thickness of the overburden h*.

Not all of the laccoliths in the Henry Mountains are circular in plan. Indeed, some are tongue-shaped masses of igneous rock that are elongated in a radial direction from the central intrusive complex forming each prominent mountain (Hunt *et al.,* 1953). These laccoliths may be fed laterally from the central igneous complex or vertically from radial dikes. Plate theory models utilizing an elliptical plan have been used by Pollard and Johnson (1973, figs 1–3) to understand the differences between the end-member cases of circular and very elongate (approximately two dimensional) laccoliths. In most respects these amount to small quantitative differences, for example in the magnitude of the deflection or stress, and in the general mechanical behavior of the model.

In the following sections the laccolith problem shown in Fig. 1 is analyzed, at first, without the assumption of a prescribed a, and then by including other improvements in the formulation, while retaining the linear bending theory of plates. The aim of this paper is to contribute to the determination of an analytically satisfactory solution for the laccolith problem under consideration, and at the same time establish the proper governing equations for problems of this type.

FORMULATION OF THE PROBLEM

To demonstrate a shortcoming of the published laccolith analyses, consider the simple two dimensional problem shown in Fig. 1. In the above references it is assumed that the mechanical properties of the overburden are represented by an elastic plate in bending that is being lifted up locally by the intruding magma. The strata under laccoliths are assumed to respond like a rigid body (compared to the flexibility of the overburden).

The corresponding formulation utilizes the simplifying assumption of cylindrical bending (plane strain), i.e. $w = w(x)$ only, and the anticipated symmetry of the bent plate. It consists of the equations

$$\left.\begin{array}{ll} Dw^{\text{iv}}(x) = q_0 - p & 0 \le x \le a \\ w(x) \equiv 0 & a \le x < \infty \end{array}\right\} \quad (1)$$

and the boundary conditions

$$\left.\begin{array}{ll} w'(0) = 0; & w'''(0) = 0 \\ w(a) = 0; & w'(a) = 0 \end{array}\right\} \quad (2)$$

where $w(x)$ is the vertical displacement of the plate reference plane at point x with positive displacements directed downward in the positive z direction, $(\ )' = \mathrm{d}(\ )/\mathrm{d}x$, D is the equivalent flexural stiffness of the overburden plate, q_0 is the constant weight of the overburden per unit horizontal area, and p is the upward magmatic pressure.

In the publications cited above it is understood that the process of intrusion involves a continuous change in horizontal dimension, as the magma spreads laterally during growth of the laccolith (see e.g. the discussion of the Black Mesa laccolith in Pollard and Johnson, 1973, fig. 25). Nonetheless, these analyses were based on the boundary conditions defined at a particular width, $x = a$, as in equation (2) above, so the width had to be specified a priori. Then, the solution to the governing equations (1) and (2) was used to compute the vertical displacement distribution of the plate reference plane over the range $0 \le x \le a$, and the associated stress distribution throughout the plate. These distributions were used to infer the mechanical state of the overburden at a time in the evolution of the laccolith when the width had reached that particu-

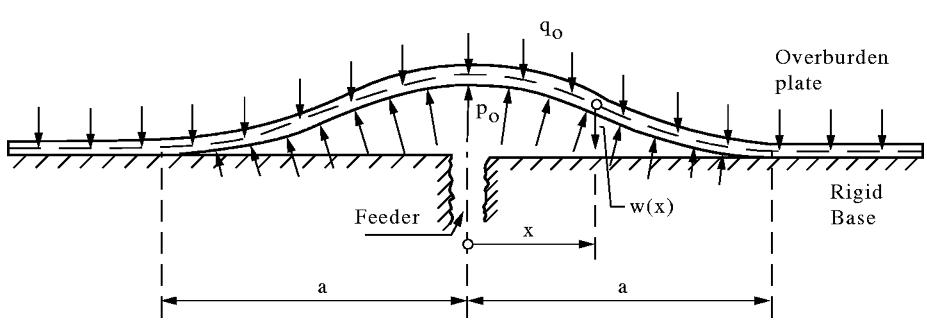

Fig. 1. Analytical model for laccolith structure under consideration.

lar value. Earlier and later times in the laccolith evolution were evaluated by specifying the appropriate width and recalculating the displacements and stresses.

This procedure does not address the question: what is the equilibrium width of the laccolith for a given magmatic pressure (or volume of magma) and resistance to bending? Indeed, from this analysis it is not obvious whether the laccolith would continue to grow in width beyond that specified by $x = a$, or if the laccolith would have stopped growing in width before reaching the specified value. Using the solution to equations (1) and (2) for a particular width, one can choose a greater or lesser magma pressure and calculate a greater or lesser maximum displacement, $w(0)$, of the plate. Presumably, for some values of pressure, the plate would lift up at the periphery $x = a$, the magma would flow laterally, and the laccolith would propagate by increasing in width. Obviously, the width has to depend on the magnitude of the pressure p, as well as on the plate (overburden) parameters.

The length $2a$ has to result from a properly posed analysis. Following Kerr (1976), who developed the variational method for structures with varying matching points, and Kerr and El-Aini (1978), the additional condition for the determination of a is

$$w''(a) = 0. \tag{2a}$$

Noting that the bending moment in the plate is given by the equation $M_{xx}(x) = -Dw''(x)$ (Timoshenko and Woinowsky-Krieger, 1959), it follows that condition (2a) implies that the bending moments M_{xx} are zero along the periphery of the laccolith. This condition appears justified mechanically, when considering Fig. 1. Namely, since the bending moments are zero in the adjoining regions (for $|x| > a$) where the plate is in contact with the flat base, hence $w \equiv w' \equiv w'' \equiv 0$ for $|x| > a$, the bending moments M_{xx} are also zero along the separation lines at $|x| = a$.

Condition (2a) does not agree with the finding by Pollard and Johnson (1973, Part II, fig. 12B), who *a priori* assumed that the laccolith width is known, assigned to it a number, and then found that $M_{xx}(a) \neq 0$.

ANALYSIS

Analysis based on the theory of plates, but without the assumption that the separation lines are a priori *prescribed*

The general solution of the differential equation in (1) for $p = p_0 = \text{constant}$ is

$$w(x) = A_1 + A_2 x + A_3 x^2 + A_4 x^3 + \frac{(q_0 - p_0)x^4}{24D} \tag{3}$$

Substitution of this expression into the boundary conditions (2) yields

$$\left.\begin{aligned} A_1 &= \frac{(a_0 - p_0)a^4}{24D}; \qquad A_2 = 0 \\ A_3 &= -\frac{(q_0 - p_0)a^2}{12D}; \qquad A_4 = 0 \end{aligned}\right\} \tag{4}$$

Thus, the solution is (Pollard and Johnson, 1973, equation 9a)

$$w(x) = \frac{(q_0 - p_0)}{24D}(a^2 - x^2)^2 \quad 0 < x < a. \tag{5}$$

If condition (2a) is imposed for the determination of a, it follows that

$$\frac{(q_0 - p_0)a^2}{3D} = 0. \tag{6}$$

Since $a \neq 0$ and $D \neq 0$, this equation is satisfied only when $p_0 = q_0$, and a is arbitrary. Thus $w(x) \equiv 0$, and there is no bending or displacement of the plate.

The above analysis does not yield a reasonable result, since it is expected that p_0 should be greater than q_0, in order to lift the plate off the base, and $w(x)$ should not be equal to zero. This is the reason why in the earlier papers mentioned above, a was assumed to be *a priori* fixed. Therefore, in the following section the above analysis is generalized.

Analysis based on the additional improvement that the magmatic pressure is not constant

In addition to dropping the assumption that a is fixed, the problem is generalized further by prescribing that the magmatic pressure varies by decreasing toward the periphery of the conduit as

$$p(x) = \mathring{p}\,[1 - (x/a)^n]. \tag{7}$$

This decrease in pressure with distance from the feeder is consistent with the known behavior of magmas. Whether the rheological properties of the magma are best described as Newtonian viscous or pseudoplastic, the drag along the wall of the conduit produces a drop in pressure, as the magma flows from the feeder toward the periphery (Johnson and Pollard, 1973). The exact form of this pressure distribution will depend upon the properties of the magma (and how it changes with shearing and heat loss along the conduit) and the changes in geometry of the conduit. As the laccolith approaches an equilibrium width, the magma will stop flowing and the pressure distribution will approach a uniform value for magmas with little or no strength. For Bingham magmas the inherent strength can support a pressure drop even if flow has ceased.

In expression (7), n is a parameter to be chosen. The distributions of p for different values of n are shown in Fig. 2. Note that $n = 1$ corresponds to the linearly varying case discussed by Pollard and Johnson (1973,

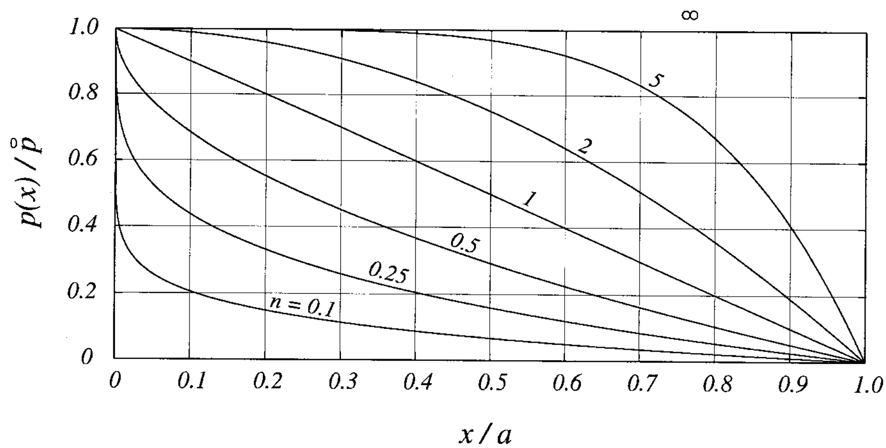

Fig. 2. Magmatic pressure according to equation (7).

Part II, fig. 7) and $n = \infty$ corresponds to a constant pressure p_0 discussed above, and used previously by Johnson (1970) and Turcotte and Schubert (1982).

It is not the purpose of this paper to investigate the rheology of magmas and how the changing shape of the magmatic conduit effects the pressure distribution; therefore, a specific n-value is not prescribed. In its general form (7), it includes many of the cases applicable to laccoliths.

In equation (7) the peak magmatic pressure $\mathring{p}$ at $x = 0$ is, as yet, unknown. Its value is needed for the assessment of the laccolith formation. Therefore, there is a need for an additional condition for the determination of $\mathring{p}$. In this connection it should be noted that field data of a known laccolith structure may directly yield an estimate of the *volume of the intrusion V*, per unit length in the y-direction (i.e. perpendicular to the plane of view in Fig. 1), but not the intensity of the magmatic pressure. For this reason, in the following analysis an *additional condition is postulated* for the determination of $\mathring{p}$, by equating the measured volume of the intruded magma, V, with the corresponding analytical expression.

Therefore, the proposed analytical formulation for the problem shown in Fig. 1 consists of the equations

$$\left.\begin{array}{ll} Dw^{\text{iv}}(x) = q_0 - \mathring{p}\,[1 - (x/a)^n] & 0 \le x \le a \\ w(x) \equiv 0 & a \le x < \infty, \end{array}\right\} \tag{8}$$

the boundary conditions

$$w'(0) = 0; \quad w'''(0) = 0; \tag{9}$$

$$w(a) = 0; \quad w'(a) = 0; \quad w''(a) = 0, \tag{10}$$

and the volumetric condition

$$-2\int_0^a w(x)\,\mathrm{d}x = V. \tag{11}$$

The negative sign is necessary because, for the sign convention used, lift-off implies $w(x) < 0$ but $V > 0$. The third equation in (10) and the one stated as equation (11) are the additional two conditions for the determination of a and $\mathring{p}$.

The general solution of the differential equation (8) is

$$w(x) = A_1 + A_2 x + A_3 x^2 + A_4 x^3 + \frac{(q_0 - \mathring{p})x^4}{24D} + \frac{\mathring{p}\, x^{n+4}}{Da^n(n+1)(n+2)(n+3)(n+4)} \quad 0 < x < a. \tag{12}$$

Substitution of this expression into equation (9) and into the first two boundary conditions in (10) yields

$$\left.\begin{array}{ll} A_1 = \dfrac{(q_0 - \mathring{p})a^4}{24D} + \dfrac{\mathring{p}\, a^4(n+2)}{2D(n+1)(n+2)(n+3)(n+4)}; & A_2 = 0 \\ A_3 = -\dfrac{(q_0 - \mathring{p})a^2}{12D} - \dfrac{\mathring{p}\, a^2}{2D(n+1)(n+2)(n+3)}; & A_4 = 0. \end{array}\right\} \tag{13}$$

The third condition in (10) yields

$$\mathring{p} = \left[\frac{(n+1)(n+3)}{(n+1)(n+3) - 3}\right] q_0. \tag{14}$$

Substitution of equation (12), in conjunction with (13), into the volumetric condition (11) results in

$$a = \left\{\frac{180\, DVn^*}{\mathring{p}\,[8n^* - 180(n+2)(n+5) + 60(n+4)(n+5) - 360] - 8q_0 n^*}\right\}^{1/5} \tag{15}$$

where

$$n^* = (n+1)(n+2)(n+3)(n+4)(n+5). \qquad (15a)$$

This completes the solution for the problem formulated by equations (8)–(11).

For example, when $n = 1$ (linear pressure drop)

$$\mathring{p} = (8/5)q_0; \quad a = (225\ DV/q_0)^{1/5}.$$

For this case, the magma pressure at the center of the laccolith, $\mathring{p}$, is greater than the overburden weight per unit area by the factor 8/5. At $x = a$ the pressure distribution described by equation (7) goes to zero, so $p(x)$ exceeds q_0 only from $x/a = 0$ to $x/a = 3/8$. The fact that the magma pressure is less than the weight per unit area of the plate from $x/a = 3/8$ to $x/a = 1$ provides the necessary conditions to reduce the curvature at the periphery to zero as required by the third boundary condition in equation (10). For a meaningful solution using this boundary condition there must be a region near the center of the plate where $p(x) > q_0$ to cause the upward displacement, and another region near the periphery where $p(x) < q_0$ to provide the downward directed force necessary to reduce the bending moment (and curvature) of the plate to zero at $x = a$. For the particular pressure distribution in equation (7) where $n = \infty$, such contrasting regions do not exist because $p(x) = p_0 =$ constant, and the only solution is the trivial one, $p_0 = q_0$, for which $w(x) = 0$ and there is no upward displacement.

According to equation (15) the laccolith length $2a$ depends, as anticipated, on the weight of the overburden q_0, the volume of the intruded magma V, the equivalent flexural stiffness of the overburden plate D, and the pressure parameter n. Also, a increases with increasing V and D, as expected. However, according to equation (14), $\mathring{p}$ depends only on q_0 and n. Although $\mathring{p} > q_0$ is intuitively correct, one would anticipate that $\mathring{p}$ should also depend on V and D.

It appears that in order to obtain an analytical solution that agrees with intuition, a further generalization of the above formulation, equations (8)–(11), is necessary. This is done in the following section.

FURTHER GENERALIZATION

Inclusion of vertical compressibility of the plate–base interface

When analyzing beams or plates that rest on a 'rigid' base and are subjected to bending, peculiar results are obtained, like the appearance of concentrated contact reactions along separation lines that do not occur in actual situations. They represent strong increases of reaction pressures in real problems, along a narrow region near the separation lines. The main cause for these problems is that the classical bending theories of beams and plates do not include deformations caused by shear. Another cause is that they tacitly imply that these beams and plates are not compressible in the transverse direction: namely that $\epsilon_{zz} \equiv 0$.

The above discussion suggests that a further generalization of the used analytical formulation should include the vertical compressibility of the plate–base contact region. This inclusion is also necessary for establishing the magmatic pressures below which the laccolith will not form; closely related to the minimum size of laccoliths, discussed by Gilbert (Johnson, 1970).

In order to maintain analytical simplicity, in the following the vertical compressibility in the contact region is represented approximately by two compressible spring layers; one attached to the bottom of the plate and the other to the top of the base, with spring coefficients k_p and k_b, respectively, as shown in Fig. 3.

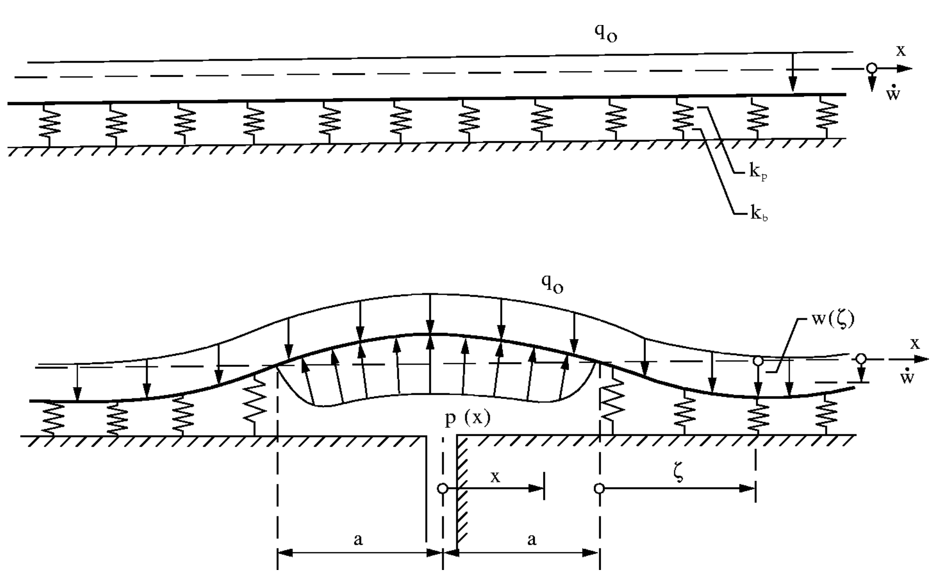

Fig. 3. Generalized model for the laccolith. (For clarity of presentation, the spring layers in the lift-off region are not shown.)

Utilizing symmetry and denoting $w(x) = w_x$ and $w(\xi) = w_\xi$, the analytical formulation of the laccolith problem shown in Fig. 3 consists of the two differential equations

$$Dw_x^{\mathrm{iv}} = q_0 - \mathring{p}\,[1-(x/a)^n] \qquad 0 < x < a \tag{16}$$

$$Dw_\xi^{\mathrm{iv}} + kw_\xi = q_0 \qquad 0 < \xi < \infty \tag{17}$$

where $k = k_\mathrm{p}k_\mathrm{b}/(k_\mathrm{p}+k_\mathrm{b})$, the boundary conditions

$$\begin{aligned} &w'_x = 0; && w'''_x(0) = 0 \\ &w_x(a) = 0; && w_\xi(0) = 0 \\ &w'_x(a) = w'_\xi(0); && w''_x(a) = w''_\xi(0) \\ &w'''_x(a) = w'''_\xi(0); && \lim_{\xi\to\infty}\{w_\xi, w'_\xi\} \to \text{finite}, \end{aligned} \tag{18}$$

and the volumetric condition

$$-2\int_0^a w_x \mathrm{d}x = V \tag{19}$$

for the determination of the eight integration constants A_1–A_8 and the two parameters $(a, \mathring{p})$.

Note that in the above formulation, the chosen location of the reference axis x of the overburden plate refers to the case when $q_0 \equiv 0$, as shown in Fig. 3.

The general solutions of equations (16) and (17) are

$$w_x = A_1 + A_2x + A_3x^2 + A_4x^3 + \frac{(q_0-\mathring{p})x^4}{24D} + \frac{\mathring{p}\,x^{n+4}}{Da^n(n+1)(n+2)(n+3)(n+4)} \tag{20}$$

and

$$w_\xi = \mathrm{e}^{-\beta\xi}(A_5\cos\beta\xi + A_6\sin\beta\xi) + \mathrm{e}^{\beta\xi}(A_7\cos\beta\xi + A_8\sin\beta\xi) + \frac{q_0}{k} \tag{21}$$

where

$$\beta = \sqrt[4]{\frac{k}{4D}}. \tag{21a}$$

Substitution of above expressions for w_x and w_ξ into equation (18), except for $w''_x(a) = w''_\xi(0)$, yields the constants

$$\begin{aligned} A_1 &= \frac{(q_0-\mathring{p})a^4}{24D} - \frac{(q_0-\mathring{p})a^2}{4\beta^2D} - \frac{q_0\beta a}{k} + \frac{\mathring{p}\,a^2[2(\beta a)^2-(n+3)(n+4)]}{4\beta^2D(n+1)(n+3)(n+4)}; \\ A_2 &= 0; \quad A_4 = 0; \quad A_5 = -\frac{q_0}{k}; \\ A_3 &= \frac{-(q_0-\mathring{p})a^2}{12D} + \frac{(q_0-\mathring{p})}{4\beta^2D} + \frac{q_0\beta}{ka} + \frac{\mathring{p}}{4\beta^2D(n+1)} - \frac{\mathring{p}\,a^2}{2D(n+1)(n+2)(n+3)}; \\ A_6 &= \frac{\mathring{p}\,a}{2\beta^3D(n+1)} + \frac{(q_0-\mathring{p})a}{2\beta^3D} + \frac{q_0}{k}; \quad A_7 = 0; \quad A_8 = 0. \end{aligned} \tag{22}$$

Substitution of (20) and (21) into the remaining condition $w''_x(a) = w''_\xi(0)$, and utilizing the determined expressions for A_3 and A_6 given in (22), yields the expression

$$\mathring{p} = \frac{2(\beta a)^3 + 6(\beta a)^2 + 6(\beta a) + 3}{(\beta a)\left\{2(\beta a)^2 + 6(\beta a) + 3 - \frac{6}{(n+1)}\left[\frac{(\beta a)^2}{(n+3)} + (\beta a) + \frac{1}{2}\right]\right\}}q_0 \tag{23}$$

For the case when the magmatic pressure is uniform, $p(x) = p_0 =$ constant. According to equation (7) and Fig. 2, this corresponds to $n \to \infty$. Thus, $p_0 = \mathring{p}$. For this case, the bracketed term in the denominator of equation (23) vanishes. The resulting relation is

$$p_0 = \frac{2(\beta a)^3 + 6(\beta a)^2 + 6(\beta a) + 3}{(\beta a)[2(\beta a)^2 + 6(\beta a) + 3]}q_0 \tag{23a}$$

or rewritten

$$p_0 = \left\{1 + \frac{3[1+(\beta a)]}{(\beta a)[2(\beta a)^2 + 6(\beta a) + 3]}\right\}q_0. \tag{23b}$$

Note that, as expected, $\mathring{p}$ and p_0 are $> q_0$, and that, unlike equation (14) where p depends only on n and q_0, $p_0 = \mathring{p}$ also depends on the overburden plate parameters and on k.

Finally, noting (20) and (22), and substituting

$$w_x = A_1 + A_3x^2 + \frac{(q_0-\mathring{p})x^4}{24D} + \frac{\mathring{p}\,x^{n+4}}{Da^n(n+1)(n+2)(n+3)(n+4)} \tag{20a}$$

into the volumetric condition (19) results in

$$\frac{2}{15}(q_0 - \mathring{p})(\beta a)^5 - (q_0 - \mathring{p})(\beta a)^3 - q_0(\beta a)^2 + \mathring{p}\,(\beta a)^3 \frac{[3(\beta a)^2 - (n+3)(n+4)]}{(n+1)(n+3)(n+4)}$$

$$- \mathring{p}\,(\beta a)^5 \frac{[(n+4)(n+5) - 6]}{(n+1)(n+2)(n+3)(n+4)(n+5)} = -3\beta^5 DV. \tag{24}$$

When the magmatic pressure is uniform, $n \rightarrow \infty$, $p_0 = \mathring{p}$, and equation (24) reduces to

$$\frac{2}{15}(q_0 - p_0)(\beta a)^5 - (q_0 - p_0)(\beta a)^3 - q_0(\beta a)^2 = -3\beta^5 DV. \tag{24a}$$

Equation (23) and (24) are the two additional relations for the determination of the laccolith width, $2a$, and the magmatic pressure $p(x) = \mathring{p}[1 - (x/a)^n]$.

This completes the analytical solution of the problem shown in Fig. 4. It consists of w_x and w_ξ given in (20) and (21), the constants in (22), and the two relations (23) and (24) for the determination of the lift-off length, $2a$ and the magmatic pressure, p.

To demonstrate the characteristic features of this solution, at first equation (23) is evaluated numerically for the n-values used in Fig. 2, the *effective* overburden thickness h = 500 m, and k = 20,000 kN m^{-3} and 50,000 kN m^{-3}. The obtained results are shown in Fig. 4.

Note that the effect of the chosen compressibilities of the overburden-base region, k, is not significant for $2a > 3$ km. Also note, that for any of the n-values used (thus, also for p = constant), a laccolith with a very small width $2a$ will exist provided $\mathring{p}$ (or p_0) is sufficiently larger than q_0. This result seems to partially explain the observation by Gilbert (1877) about the smallest size of encountered laccoliths (Johnson, 1970, Chapter 2). When considering this aspect, the interpretation of the 'effective overburden thickness' of each layer, h, by Pollard and Johnson (1973) should be taken into consideration for the small width of the laccolith, $2a$. Also note, that below a certain size the pressure required to bend the overburden is excessive. Therefore, a possible explanation of this phenomenon is that, at first, the magma is injected as a sill that locally deforms both overlying and underlying strata.

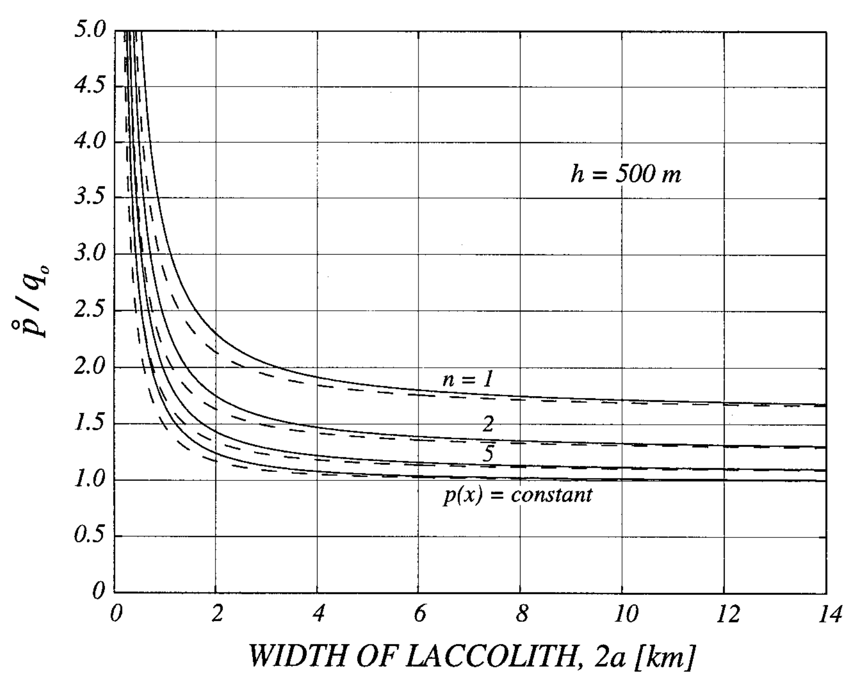

Fig. 4. Dependence of $2a$ on $\mathring{p}/q_0$, n, and k. (—- k = 20,000 kN m^3; —— k = 50,000 kN/m^{-3}.)

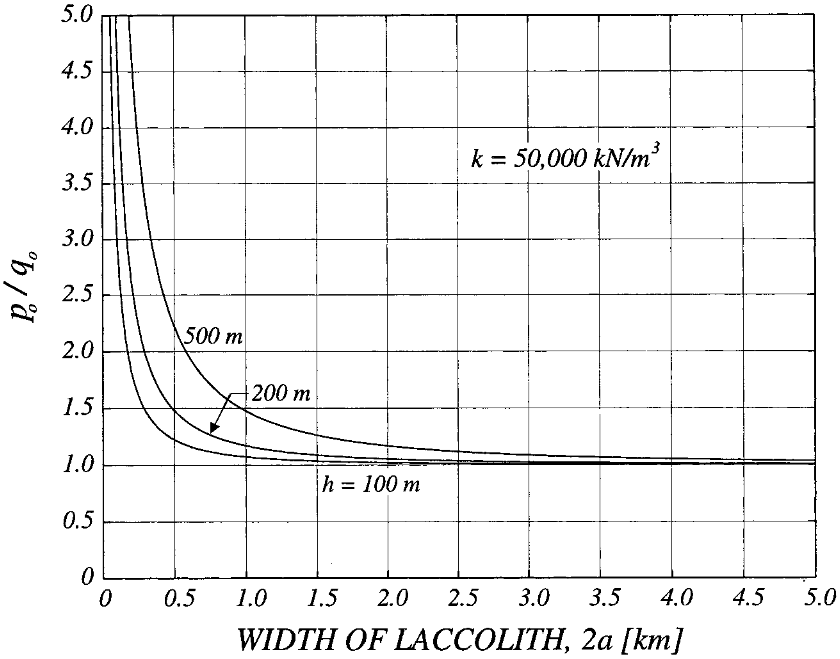

Fig. 5. Dependence of $2a$ on p_0/q_0 and the overburden thickness h.

According to equation (23), as $\beta a \rightarrow \infty$

$$\frac{\overset{\circ}{p}}{q_0} = \frac{1}{1 - 3/[(n+1)(n+3)]}. \tag{25}$$

This equation establishes the horizontal asymptotes for the curves in Fig. 4. For the case when p = constant = p_0 (i.e. when $n = \infty$) then, according to equation (25), $p_0 = q_0$. This is anticipated, because of the vertical equilibrium of the entire overburden plate. For variable p, according to equation (25), $\overset{\circ}{p} > q_0$. This condition also agrees with vertical equilibrium.

To show the influence of the effective overburden thickness h, equation (23) is evaluated for $p = p_0$ = constant ($n = \infty$), k = 50,000 kN m^{-3}, and h = 100, 200, and 500 m. The results are shown in Fig. 5. This graph shows the dependence of the lacco-

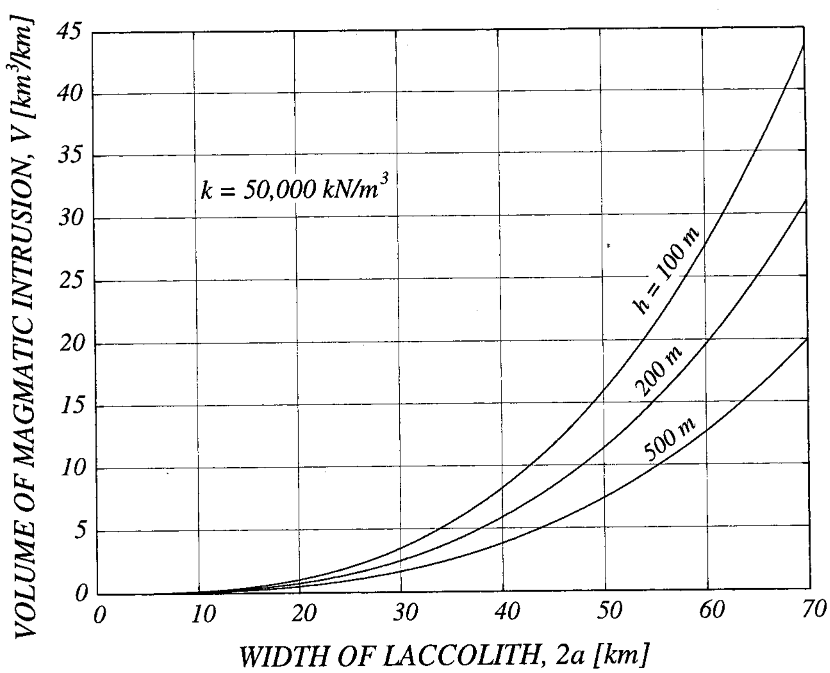

Fig. 6. Dependence of $2a$ on the volume of intruded magma, V.

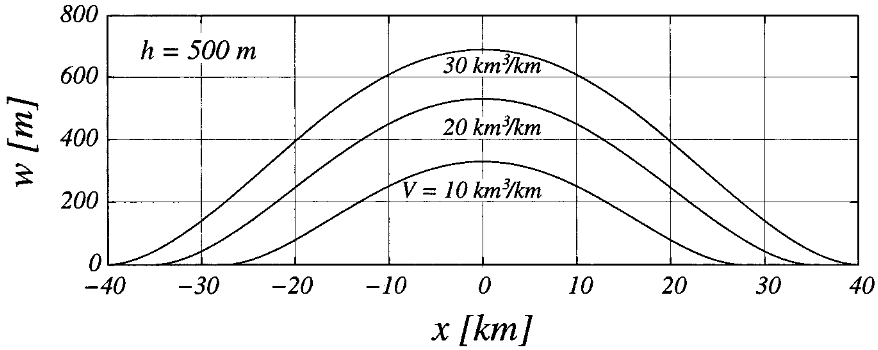

Fig. 7. Dependence of laccolith profile on volume, V.

lith size on its stratigraphic position (Johnson, 1970, p. 70); the thicker the overburden h the larger the size of the laccolith $2a$.

To determine the effect of the volume of the intruded magma, V, on the lift-off width of the laccolith, $2a$, the pressure $\mathring{p}$ (or p_0) given in equation (23) (or 23a) has to be substituted into equation (24) (or 24a). For the case of uniform pressure $p_0 = \text{constant}$, equation (23b) is substituted into equation (24a). The resulting relation is

$$\frac{(\beta a)^3[2(\beta a)^2 + 12(\beta a) + 15]}{5[2(\beta a)^2 + 6(\beta a) + 3]} = \frac{3\beta^5 DV}{q_0}. \quad (26)$$

Equation (26) was evaluated numerically, by assuming values of a and β and then by calculating the corresponding V-values from equation (26). The graphs for $h = 100$, 200, and 500 m, and $k = 50{,}000$ kN m^{-3} are shown in Fig. 6.

Note the dependence of the laccolith size on its stratigraphic position and the volume of the magmatic intrusion. For example, for $V = 10$ km^3 and $h = 100$ m the width of the laccolith is $2a \cong 43$ km, whereas for $h = 500$ m it is $\cong 55$ km.

It is also of interest to determine the effect of the volume of the intruded magma V on the shape of the deformed overburden 'plate'. As an example, equation (20) was evaluated, noting the integration constants in equation (22), and the determined a vs V values in Fig. 6. The obtained results are shown in Fig. 7. Note that a will be smaller if the spring layers are assumed attached to each other and are then subjected to a 'separation' criterion.

As expected, for an overburden plate of given thickness, the larger the volume of the intruded magma V, the larger the width of the laccolith $2a$ and its height.

CLOSELY RELATED PROBLEM

A generalization of the laccolith problem shown in Fig. 1 was discussed by Petraske *et al.* (1978).

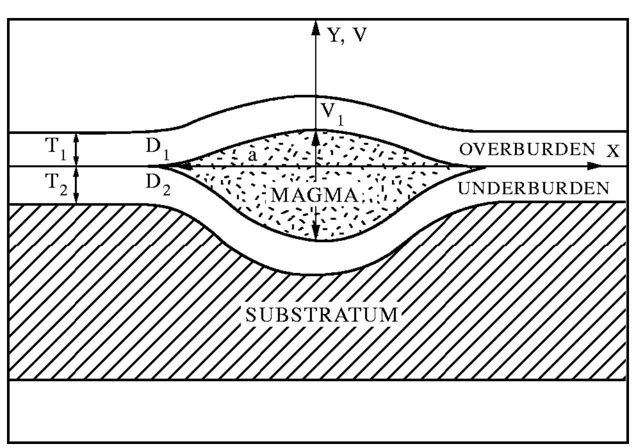

Fig. 8. Model for analyzing the effect of magma intruded into lithosphere, according to Petraske *et al.* (1978).

According to this paper, field studies of deformation about some basic intrusives indicate that bending of the strata below an intrusion, as well as above, must be considered. Therefore, under certain geologic conditions the floor of the magma chamber may not be assumed to be 'rigid', since it will be deformed as well by the intruding magma. Following the concepts of plate tectonics, they assumed the model shown in Fig. 8 for the analytical study of the emplacement mechanics of basic intrusions.

In this idealized model the substratum is the asthenosphere which acts like a weak fluid. The material below the intrusion and above the substratum is referred to as the 'underburden'. The magma is assumed to intrude horizontally into a lithosphere which has homogeneous and isotropic material properties. The magma is considered to act as a weak Newtonian fluid and the pressure was assumed to be constant throughout the formation.

Although this model may be applicable to certain geophysical situations, its analytical formulation as presented by Petraske *et al.* (1978), is not correct. For example, the differential equations for the overburden and underburden cannot be solved independently as done in this paper, since they interact along the separation lines, at $|x| = a$, and the pressure exerted by the magma on the underburden depends on both deflections. The interaction should be described analytically at $|x| = a$ by the proper number of matching conditions. For example, at $|x| = a$ the deflections of both plates should be the same (not necessarily = 0). Also, at $|x| = a$ the slopes of the upper and lower plates should be equal, since bending plate theory is being utilized.

The authors did not use matching conditions, but rather a number of boundary conditions of questionable validity. For example, there is no justification to assume that at the ends of the intrusion, at $|x| = a$, the overburden deflection V_1 is = 0 and the slopes $\mathrm{d}V_1/\mathrm{d}x = 0$. Nor is there justification to state that along $|x| = a$ the deflection of the underburden V_2 is = 0. Also, the assumption that the laccolith length $2a$ is fixed *a priori* is not justified. This value should result from a proper formulation, as discussed previously.

Therefore, the relevant conclusions arrived at by the authors, regarding the formation of the laccolith as shown in Fig. 8, should be used with caution.

CONCLUSIONS

In the present paper, a sequence of improvements was introduced in the analysis of the laccolith shown in Fig. 1. From the obtained results it may be concluded that: (1) The width of the laccolith should not be assumed to be *a priori* fixed. It should be determined from an additional matching condition at the separation lines $|x| = a$. For a justification of this approach using variational calculus, refer to Kerr (1976, Problem III) and Kerr and El-Aini (1978); (2) To obtain intuitively meaningful analytical results, the effect of vertical compressibility of the interface region, between the overburden plate and the upper layer of the 'rigid' base, has to be included. In the present paper this was modeled, approximately, by including elastic spring layers in the contact region; (3) The magmatic pressure p, which enters the governing differential equation for the overburden 'plate', is at first unknown. However, since the volume of the magma intrusion V may be estimated from field data of a known laccolith structure, an additional condition was postulated for the determination of $\mathring{p}$ (or p_0), by equating the measured V with the corresponding analytical expression.

The results obtained, by including these analytical generalizations, appear to satisfy many intuitive expectations. Thus, equations (16)–(19) form a reasonable analytical formulation for the laccolith problem under consideration.

It should be noted, however, that the presented analyses are based on the assumption that the laccolith structure deforms elastically and that the vertical overburden displacements are relatively small. Therefore, they are valid only for the initial stages of the laccolith formation. Analyses for the later stages will have to be based on overburden equations for larger displacements, that will be able to predict also the generated axial forces caused by vertical lift-off. They will also have to utilize elastoplastic and viscoelastic constitutive equations, and related elements of fracture mechanics.

Acknowledgements—The authors wish to thank Alexander Babinski and Eric S. Carr, students at the University of Delaware, for their assistance in preparing the graphs presented in this paper.

REFERENCES

Corry, C. E. (1988) *Laccolith; Mechanics of Emplacement and Growth*, The Geological Society of America Special Paper, **220**.

Gilbert, G. K. (1877) *Report on the Geology of the Henry Mountains*. U.S. Geographical and Geological Survey of Rocky Mountain Region.

Hunt, C. B., Averitt, P. and Miller, R. L. (1953) *Geology and Geography of the Henry Mountains Region, Utah*. U.S. Geological Survey Professional Paper no. **228**.

Jackson, M. D. and Pollard, D. D. (1988) The Laccolith-Stock controversy: new results from the Southern Henry Mountains, Utah. *Geological Society of America Bulletin* **100,** 117–139.

Jackson, M. D. and Pollard, D. D. (1990) Flexure and faulting of sedimentary host rocks during growth of igneous domes, Henry Mountains, Utah. *Journal of Structural Geology* **12,** 185–206.

Johnson, A. M. (1970) *Physical Processes in Geology*. Freeman, Cooper & Co, San Francisco.

Kerr, A. D. (1976) On the derivation of well-posed boundary value problems in structural mechanics. *International Journal of Solids and Structures* **12,** 1–11.

Kerr, A. D. and El-Aini, Y. M. (1978) Determination of admissible temperature increases to prevent vertical track buckling. *Journal of Applied Mechanics* **45,** 565–573.

Petraske, A. K., Hodge, D. S. and Shaw, R. (1978) Mechanics of emplacement of basic intrusions. *Tectonophysics* **46,** 41–63.

Pollard, D. D. and Johnson, A. M. (1973) Mechanics of growth of some laccolith intrusions in the Henry Mountains, Utah, II; bending and failure of overburden layers and sill formation. *Tectonophysics* **18,** 311–354.

Turcotte, D. L. and Schubert, G. (1982) *Geodynamics: Application of Continuum Physics to Geological Problems*. John Wiley & Sons, New York.